Upcrossing Inequalities for Powers of Nonlinear Operators and Chacon Processes

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Abstract

In this paper we use a filling scheme technique to prove the integrability of two counting functions. The integrability of one of these functions implies the a.e. convergence for the powers $T^n f(x)/n$ where T belongs to a certain class of nonlinear operators. The other counting function generalizes the upcrossing function considered by Bishop to the case of Chacon processes. In the last section we prove the connection between our results and previous results by Bishop. We also provide a result which connects upcrossings and oscillations.

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1 Introduction

References [2], [3], [4], introduced the notion of upcrossing inequalities (u.i.) in the Ergodic Theory setting. We recall the reader that the upcrossing function counts the number of times a given sequence oscilates through a given interval. Roughly speaking, these results established an integral inequality for a counting function relevant to some convergence problem. Once this inequality is established, a.e. convergence follows easily; in general there is no need to appeal to the Banach principle. Recently, new attention has been given to this phenomena (see for example references [1], [8], [6]) and it is natural to investigate how fundamental are the counting methods implicit in the u.i. in the context of Ergodic Theory as well as in the broader context of Analysis.

In the next section of this paper we study the u.i. in the setting of nonlinear operators. The motivation behind this work is to prove that an u.i. holds for the nonlinear averages ([9] , [7]) and, therefore, to find a different proof for the important new result in [10] which establishes a.e. convergence for nonlinear averages. We are able to establish a partial result in this direction, namely Theorem 2.9, which proves that the function defined as the number of times at which the powers $T^n f(x)/(n+1)$ are bigger than a certain fixed number η is integrable for a certain class of nonlinear operators T. We will indicate how this result is related to the main result in [10] . It is an open problem to extend Theorem 2.9 in order to prove the a.e. convergence of the nonlinear averages considered in that reference.

In the third section we will extend the results in [1], that reference gives a new proof of the u.i. in the context of the Chacon-Ornstein ratio theorem. One of the snapshots of that proof is the derivation of a connection between the filling scheme technique and the u.i. We will extend this technique in order to deal with the case of bounded Chacon processes. This case is neither covered in [1] nor by the more general setting developed by Bishop in [3] and [4]. In this way we obtain an improved u.i. This technique allows us to further exploit the symmetry present in ratio theorems, for instance, we derive a surprising bound (independent of the function) for the *fluctuations* of ergodic averages.

In the last section we show how the upcrossing inequality obtained in section 3 implies the main combinatorial inequality from [2]. We also provide a simple argument to derive bounds for the integral of the *oscillations* of ergodic averages from upcrossing inequalities.

To prove our main results we will use modified versions of the techniques used in [4] and [1]. These two techniques are analogous; in the first paper an inductive procedure indexed by the upcrossing times is defined; the thrust of that paper is then to show how this inductive scheme is related to a "filling property." Conversely, in the second reference, an inductive procedure indexed by the filling times is defined. Thus the proof consists in relating these times to the upcrossing times.

In the sequel $(X, \mathcal{F}, \mu)$ will be a complete σ-finite measure space (unless explicitly stated otherwise); we will study $L^1(X, \mathcal{F}, \mu)$, and will denote this space more briefly by L^1.

2 Nonlinear Powers

We introduce some notation and definitions which will be special for this section. In particular, for simplicity, X will be a finite measure space. We will consider a fixed map T from L^1 to itself (i.e. a mapping on L^1).
T is called *order preserving* if

(2.1) $$\text{f} \leq g \quad \text{then} \quad Tf \leq Tg \quad (f, g \in L^1).$$

We say that T is L^p *nonexpansive* if

(2.2) $$\|Tf - Tg\|_p \leq \|f - g\|_p \quad (f, g \in L^P).$$

We say T is L^p *norm decreasing* if

(2.3) $$\|Tf\|_p \leq \|f\|_p \quad (f \in L^P).$$

The following theorem is proved in [10].

Theorem 2.4. *Let T be an order preserving, L^1 and L^∞ nonexpansive mapping on L^1. Then for any $f \in L^1$ the sequence $T^n f/(n+1)$ is almost everywhere convergent.*

In the context of order preserving, L^1 and L^∞ nonexpansive maps on L^1 Theorem 2.4 is equivalent to the a.e. convergence of the nonlinear Cesaro averages. To define these averages we first define the nonlinear partial sums:

$$S_0 f = f, \quad S_{n+1} f = f + TS_n f \quad (n \geq 0) \tag{2.5}$$

Then the associated Cesaro averages are defined by

$$A_n f = S_n f/(n+1) \tag{2.6}$$

To show that Theorem 2.4 is equivalent to a.e. convergence for the non-linear averages we argue as follows. Given T as above and $f \in L^1$ the mapping defined by $T_f g = Tg + f$ satisfies the same properties as T and $T_f^{n+1} f = S_n f$. Therefore the convergence of $T_f^n f/n+1$ implies the convergence of $A_n f$. Conversely, defining $T_f g = Tg - f$, it follows that $S_{f,n} Tf = T^{n+1} f$ where $S_{f,n}\, g$ are the partial sums in (2.5) but with respect to T_f. Hence, the convergence of $A_n f$ implies the convergence of $T^n f/(n+1)$.

In this section we will prove the following theorem.

Theorem 2.7. *Let T be an order preserving map on L^1 which is L^1 nonexpansive, L^∞ norm decreasing and which satisfies $T0 = 0$, then:*

$$\lim_{n\to\infty} \frac{T^n f(x)}{(n+1)} = 0 \quad a.e.$$

for all $f \in L^1$.

Remarks

i) Due to the assumption that T is L^∞ norm decreasing, the relation with nonlinear averages mentioned below equation (2.6) is no longer valid. This is consistent with the results from [9] and [10]. In these references it is proven that $A_n f$ does not need to converge if the assumption that T is L^∞ *nonexpansive* in Theorem 2.4 is replaced by the hypothesis that T is L^∞ *norm decreasing.* We refer to reference [9], Proposition 5.1, for an example in which the hypothesis of Theorem 2.7 hold, but the hypothesis of Theorem 2.4 are not satisfied.

ii) If $f \in L^1$ is assumed to be nonnegative, the condition of T being L^∞ norm decreasing can be relaxed to $T\alpha \leq \alpha$.

Theorem 2.7 will follow easily from the next theorem. Before stating this result we need to introduce the counting function related to the convergence of $T^n f/(n+1)$.

Definition 2.8. *For fixed $x \in X$, a natural number $N \geq 0$ and real $\eta > 0$ we define $\omega_N(x)$ as the maximum integer n for which there are n numbers v_i which satisfy, $0 \leq v_1 < v_2 < \ldots < v_n \leq N$ and $T^{v_i} f(x) \geq (v_i + 1)\eta$ for all i=1, ... ,n. The numbers $v_1, v_2, \ldots, v_{\omega_N(x)}$ will be called the sequence (see remark below) of maximal times at x. We also define:*

$$\omega(x) = \sup\{\omega_N(x) : N \geq 0\}.$$

Remark

It is clear that it is possible to choose, for each x, a unique sequence of integers v_n independently of N. Namely, let v_1 be the smallest integer such that $T^{v_1}f(x) \geq (v_1+1)\eta$. Then, given v_i, define v_{i+1} to be the smallest number greater than v_{i+1} which satisfies $T^{v_{i+1}}f(x) \geq (v_{i+1}+1)\eta$.

Theorem 2.9. *Let T be as in Theorem 2.7 and $f \in L^1$, $f \geq 0$. For each N and η as above,*

$$\int \omega_N \, \eta \, d\mu \leq \int f \, d\mu. \tag{2.10}$$

Hence

$$\int \omega \, \eta \, d\mu \leq \int f \, d\mu. \tag{2.11}$$

Proof. The proof of Theorem 2.9 will be completed by establishing several properties of the following construction. Fix an integer $N \geq 0$ and a real number $\eta > 0$. We will inductively define nonnegative functions f_n, d_n, e_n, φ_n and π_n where $0 \leq n \leq N$. For each fixed x these functions will be constructed in two stages plus some initial conditions. The different stages will be defined through the maximal times.

Initial conditions: The following definitions hold for all points of X.

$$\varphi_0 = f, \quad d_0 = 0, \quad \pi_0 = \eta.$$

For a fixed x and $n \geq 0$ define.

First Stage: The number n satisfies $n \neq v_i$ for all $i \in \{1, \ldots, \omega_N(x)\}$. In this case we say that n is of *positive type* (for the given x). Define,

$$e_n(x) = \varphi_n(x) \wedge \pi_n(x) \quad \pi_{n+1}(x) = \pi_n(x) - e_n(x).$$

Second Stage: The number n equals v_i for some $i \in \{1, \ldots, \omega_N(x)\}$. In this case we say that n is of *transition type* (for the given x). Define,

$$e_n(x) = \pi_n(x) \quad \pi_{n+1}(x) = \eta.$$

For all x we set,

$$\varphi_{n+1} = T^{n+1}f - d_{n+1} \text{ and } \quad d_{n+1} = T(d_n + e_n).$$

It is convenient to set $f_n = \varphi_n - e_n$. We then obtain that for all $n = 0, 1, \ldots, N$

$$\sum_{k=0}^{n} e_k = \omega_n \eta + (\eta - \pi_{n+1}). \tag{2.12}$$

This equation is easily checked for $n = 0$ by noticing that $\omega_0(x) = 0$ if $n = 0$ is of positive type and $\omega_0(x) = 1$ otherwise. Assuming (2.12) is valid for n, then we will prove that it is also true for $n + 1$. If $n + 1$ is of positive type, we conclude

$$\omega_{n+1}(x) = \omega_n(x), \quad \pi_{n+2}(x) = \pi_{n+1}(x) - e_{n+1}(x).$$

This implies

$$\sum_{k=0}^{n+1} e_k = e_{n+1} + \omega_{n+1}\eta + (\eta - \pi_{n+2} - e_{n+1}).$$

If $n + 1$ is of transition type

$$\omega_{n+1}(x) = \omega_n(x) + 1, \quad \pi_{n+2}(x) = \eta, \quad e_{n+1}(x) = \pi_{n+1}(x).$$

Hence

$$\sum_{k=0}^{n+1} e_k = \omega_{n+1}\eta = \omega_{n+1}\eta + (\eta - \pi_{n+2}).$$

Therefore we have established (2.12).

The next step is to prove that the functions $f_n, \quad \varphi_n, \quad e_n, \quad and \quad \pi_n$ are nonnegative functions in L^1. The fact that these are functions in L^1 follows from induction and recalling that T is an operator on L^1. Observe that the functions are nonnegative for $n = 0$. Assume that the statements hold for $n \geq 0$. From the definitions we immediately obtain $\pi_{n+1} \geq 0$. Moreover, we can write

$$T^n f = f_n + d_n + e_n$$

hence by the inductive hypothesis and the fact that T is order preserving,

$$T^{n+1} f \geq T(d_n + e_n) = d_{n+1}.$$

Therefore

$$\varphi_{n+1} \geq 0 \;\; and \quad e_{n+1} \geq 0.$$

It remains to prove $f_{n+1} \geq 0$. In the case $n + 1$ is of positive type for x we have $f_{n+1}(x) = \varphi_{n+1}(x) - e_{n+1}(x) \geq 0$ from the definitions. When $n + 1$ is of transition type for x we will assume $f_{n+1}(x) < 0$ and obtain a contradiction. The fact that n+1 is a transition integer for x implies

$$\eta \quad (n + 2) \leq T^{n+1} f(x) < e_{n+1}(x) + d_{n+1}(x). \tag{2.13}$$

Notice

$$0 \leq \pi_n \leq \eta, \quad 0 \leq e_n \leq \eta.$$

Given these equations, the hypothesis on T imply $d_1 \leq \eta$ and it follows by induction that $d_n \leq n\eta$ for any $n \geq 0$. Hence $e_n + d_n \leq (n + 1)\eta$ is valid for all n. This inequality combined with equation (2.13) results in a contradiction.

The last property that we need from our construction is:
For all $n = 0, 1, \dots, N$

$$\int \omega_n \, \eta \, d\mu \le \sum_{k=0}^{n} \int e_k d\mu \le \int f \, d\mu. \tag{2.14}$$

The first inequality follows from (2.12). The second inequality follows easily by induction once we prove the following inequality

$$\int \varphi_{k+1} d\mu \le \int \varphi_k d\mu \; - \int e_k d\mu \tag{2.15}$$

for all $k = 0, 1, \dots, N-1$. Using $f_n \ge 0$, $\varphi_n \ge 0$ and the fact that T is L^1 nonexpansive we can compute as follows:

$$\int \varphi_{k+1} d\mu = \int T^{k+1} f - T(d_k + e_k) d\mu \le \int |T^k f - d_k - e_k| d\mu = \int (\varphi_k - e_k) d\mu.$$

To conclude the proof of the Theorem 2.9 we notice that (2.14) proves equation (2.10). Equation (2.11) follows by Lebesgue monotone convergence theorem. □

We are now ready to proceed with the proof of Theorem 2.7.

Proof. First assume $f \ge 0$, for a fixed real number η, the function $\omega(x)$ in Theorem 2.9 counts the number of times that $T^n f(x)/(n+1)$ is greater or equal η. This function, being integrable, is finite a.e. To handle the case of an arbitrary $f \in L^1$ we introduce a new operator S on L^1 defined by

$$Sg = -T(-g).$$

Then,

$$S^n(g) = -T^n(-g). \tag{2.16}$$

The transformation S satisfies the same properties than T. Write $f = f_+ - f_-$, therefore using equation (2.16) with $g = f_-$ and the result for nonnegative functions we obtain:

$$\lim_{n\to\infty} \frac{T^n(-f_-)(x)}{(n)} = - \lim_{n\to\infty} \frac{S^n f_-(x)}{(n)} = 0 \tag{2.17}$$

Using the fact that T is order preserving,

$$T^n(-f_-) \le T^n(f) \le T^n(f_+).$$

Dividing by n, taking $n \to \infty$, using equation (2.17) and the result for nonnegative functions the proof is complete. □

3 Upcrossings fo Chacon Processes

As it was pointed out in the introduction, the main purpose of this section is to generalize the results in [1] to the setting of bounded Chacon Processes. These processes are not included in the framework considered by Bishop. The key element of the proof is Lemma 3.30. By using this result we are able to make a distinction between the *generalized upcrossing function* and the *fluctuation function* and in this way to further exploit the symmetry present in ratio theorems. As a corollary we obtain the surprising result that the integral of the fluctuation function for the ergodic averages has a bound which is independent of the function. The proof consists in an extension of the techniques used in [1], hence a certain amount of repetition will occur. Henceforth all the operators considered will be linear. We will study upcrossings, which, we recall, are defined as follows: given a sequence $\{a_r\}_{r\geq 0}$ of real numbers and numbers $\alpha < \beta$, the upcrossings of the sequence through the interval $[\alpha, \beta]$ is defined by

$$w = \sup\{k : \zeta = (u_i, v_i)_{i=1,\dots,k.} \ \& \ u_i < v_i < u_{i+1} \text{ for } i = 1, \dots, k\}$$

where ζ satisfies $a_{u_r} \leq \alpha$ and $a_{v_r} \geq \beta$ for all $r = 1, \dots, k$. We warn the reader that the word upcrossings is sometimes used for quantities that bound from above the upcrossings defined above. Later in this section we will redefine the upcrossings, but this new definition will provide an upper bound for this original definition.

Definition 3.1. *A collection $\{d_0, d_1, \dots\}$ of $(X, \mathcal{F}, \mu)$-measurable, real valued, nonnegative functions is said to be a Chacon (or admissible) process (with respect to a nonnegative, linear, L^1-contraction) if:*

$$Td_i \leq d_{i+1} \quad holds\ for\ all \quad i \geq 0,$$

where we have implicitly extended the action of T from L^1 to include nonnegative measurable functions. If, in addition, the partial sums defined by:

$$D_n = d_0 + d_1 + \dots + d_n$$

satisfy

$$\gamma(D) \equiv \sup_{n\geq 1} \frac{1}{n} \int_\Omega D_n d\mu < \infty$$

then the process is called bounded. $\gamma(D)$ is called the time constant of the process.

Definition 3.2. *A mapping*

$$T : L_1(X, \mathcal{F}, \mu) \to L_1(X, \mathcal{F}, \mu)$$

is called Markovian if T is a positive linear operator satisfying

$$\int_X f \, d\mu \ = \ \int_X Tf \, d\mu \qquad \forall f \in L_1^+ \ .$$

We will study the a.e. convergence of the ratios

$$R_n(x) = \frac{D_n(x)}{C_n(x)}, \qquad D_n(x) = \sum_{k=0}^{n} d_k(x), \qquad C_n(x) = \sum_{k=0}^{n} c_k(x) \tag{3.3}$$

where $\{d_k\}$ *and* $\{c_k\}$ are Chacon processes with respect to the same operator T. In general, for the a.e. convergence of (3.3), we will need to require that T is Markovian ($T = 0$ is a counterexample) but we will assume this only when it is needed. Therefore, T will be a nonnegative L^1 contraction unless explicitly stated otherwise. Notice that given three indices $-1 \leq u_1 < v_1 < u_2$, a fixed pair of real numbers $0 \leq \alpha < \beta$ and if the following inequalities hold:

$$\frac{D_{u_i}(x)}{C_{u_i}(x)} \leq \alpha \tag{3.4}$$

$$\frac{D_{v_1}(x)}{C_{v_1}(x)} \geq \beta \qquad i = 1, 2. \tag{3.5}$$

Then the following inequalities hold

$$D_{v_1}(x) - D_{u_1}(x) - (\alpha\, C_{v_1}(x) - \alpha\, C_{u_1}(x)) \geq \eta\, C_{v_1}(x) \tag{3.6}$$

$$D_{v_1}(x) - D_{u_2}(x) - (\alpha\, C_{v_1}(x) - \alpha\, C_{u_2}(x)) \geq \eta\, C_{v_1}(x) \tag{3.7}$$

where $\eta = \beta - \alpha$. Taking into account that a Chacon process multiplied by any non-negative real number still gives a Chacon process, we see that the the processes $\{\alpha c_k\}$ and $\{\eta c_k\}$ are also Chacon processes. This fact allows us to talk only about processes in Definition 3.8 below without mentioning the numbers α and β. Moreover, equations (3.6) and (3.7) show that in order to bound from above the number of where (3.4) and (3.5) hold for a given x (i.e. the upcrossings through the interval $[\alpha, \beta]$) it is enough to consider the counting function defined below in (3.13). First we need:

Definition 3.8. *Let $\{d_k\}, \{q_k\}$ and $\{p_k\}$ be three Chacon processes with respect to a positive linear contraction T. Also fix an integer $N \geq 0$. Let u_r, v_r, $r = 1, 2, \ldots,$ be elements in $\{-1, 0, 1, \ldots, N\}$, such that*

$$u_r < \quad v_r \tag{3.9}$$

$$v_r \leq \quad u_{r+1} \tag{3.10}$$

for all r. We denote a sequence $(u_r, v_r)_{r=1,2,\ldots}$ with these characteristics by ζ. For a given point x , the sequence ζ will be called a sequence of generalized upcrossing times at x for $\{d_k\}$, $\{q_k\}$ and $\{p_k\}$ if

$$D_{v_r}(x) - D_{u_r}(x) - (Q_{v_r - 1}(x) - Q_{u_r}(x)) \geq P_{v_r}(x) \tag{3.11}$$

for all r, and

$$D_{v_r}(x) - D_{u_{r+1}}(x) - (Q_{v_r-1}(x) - Q_{u_{r+1}}(x)) \geq P_{v_r}(x) \tag{3.12}$$

for all r such that u_{r+1} is defined.

The word "generalized" refers to the fact that the v-index in equations (3.6) and (3.7) has been replaced by the index $v-1$ in equations (3.11) and (3.12), there is also greater flexibility by allowing a non-strict inequality in (3.10). It is easily seen that this represents a more general definition than the usual upcrossings considered by Bishop (see ([1]). We will exploit this extra generality in the next section (see Theorem 4.4).
For a fixed x define:

$$\omega_N(x) = \sup\{k : \zeta = (u_r, v_r)_{r=1,2,\ldots,k.}\} \tag{3.13}$$

$$\chi_N(x) = \sup\{\max(k-1, 0) : \zeta = (u_r, v_r)_{r=1,2,\ldots,k.}\} \tag{3.14}$$

Where the supremum is taken over the finite set of all sequences of generalized upcrossing times at x. We call ω_N the *generalized upcrossing function.* The function χ_N is merely another way of counting oscillations and because of its clear relationship to the counting function used in [8], we call it the *fluctuation function.*

Remarks

Consider $\{d_r\}$ and $\{c_r\}$ to be as in equation (3.3), and define$\{q_r = \alpha c_r\}$ and $\{p_r = \eta c_r\}$ then the function ω_N is an upper bound for the upcrossings of the ratios (3.3) through the interval $[\alpha, \alpha+\eta]$.
The first result of this section is the following theorem.

Theorem 3.15. *Given three Chacon processes $\{d_k\}, \{q_k\}$ and $\{p_k\}$ with respect to a Markovian transformation T then:*
For each $N \geq 0$,

$$\int \omega_N \, p_0 \, d\mu \leq 2\,\gamma(D) \tag{3.16}$$

$$\int \chi_N \, p_0 \, d\mu \leq 2\,\gamma(Q) \tag{3.17}$$

where (3.16) or (3.17) hold if $\{d_k\}$ or $\{q_k\}$ are bounded processes respectively.

Remarks

From the above theorem, convergence a.e. in appropiate sets for the ratios (3.3) can be established by standard arguments ([3], [4]). Roughly speaking, in equations (3.16) and (3.17), the integrals of the oscillations of the ratios are bounded by the time constants of each process respectively. The difference between both equations is the following: while the hypothesis $\gamma(D) < \infty$ gives convergence of (3.3) in $\mathbf{R}$, the condition $\gamma(Q) < \infty$ proves that the fluctuation function is finite a.e., however the ratios (3.3) may still diverge to ∞. The bound given in equation (3.16) can be improved, at least for the case where $d_k = T^k d_0$ and T is induced by a measure preserving transformation in a finite measure space. This is done in [8].

As it was shown in section 2, we now proceed to define a filling scheme which will be the key to the proof. In contrast to the filling scheme in section 2 this construction is not indexed by "upcrossing times," but "filling times," internal to the construction, are defined. We refer to [1] for a motivation of the functions defined below.

It should be noted that we construct $\varphi_{n+1}, \theta_{n+1}, \pi_{n+1}, \kappa_{n+1}, \rho_{n+1}, \sigma_{n+1}$ during step n, since these functions refer to conditions just before step $n+1$. We also construct functions e_n, h_n, f_n, g_n during step n. These functions describe results of operations during step n. The functions ρ_n, σ_n are measurable functions taking nonnegative integer values (the filling times).

Given three Chacon process, we can define the following construction:

Initial Definitions

Let

$$\varphi_0 = d_0,\ \theta_0 = q_0,\ \pi_0 = \kappa_0 = p_0,\ \rho_0 = \sigma_0 = 0.$$

Step n

Suppose that we have already defined $\varphi_n, \theta_n, \pi_n, \kappa_n, \rho_n, \sigma_n$ for some $n = 0, 1, \ldots$.

First Turn

On $\{\rho_n = \sigma_n\}$ let $e_n = \varphi_n \wedge \pi_n$. On $\{\rho_n = \sigma_n\}^c$ let $e_n = 0$. On $\{\rho_n = \sigma_n\} \cap \{e_n = \pi_n\}$ let $\rho_{n+1} = \rho_n + 1$, $\pi_{n+1} = p_0$. On $(\{\rho_n = \sigma_n\} \cap \{e_n = \pi_n\})^c$ let $\rho_{n+1} = \rho_n$, $\pi_{n+1} = \pi_n - e_n$.

Second Turn

On $\{\rho_{n+1} = \sigma_n + 1\}$ let $h_n = \theta_n \wedge \kappa_n$. On $\{\rho_{n+1} = \sigma_n + 1\}^c$ let $h_n = 0$. On $\{\rho_{n+1} = \sigma_n + 1\} \cap \{h_n = \kappa_n\}$ let $\sigma_{n+1} = \sigma_n + 1$, $\kappa_{n+1} = p_0$. On $(\{\rho_{n+1} = \sigma_n + 1\} \cap \{h_n = \kappa_n\})^c$ let $\sigma_{n+1} = \sigma_n$, $\kappa_{n+1} = \kappa_n - h_n$.

Third Turn

Let $f_n = \varphi_n - e_n$, $g_n = \theta_n - h_n$, and define

$$\varphi_{n+1} = d_{n+1} - \sum_{k=0}^{n} T^{n+1-k} e_k$$

$$\theta_{n+1} = q_{n+1} - \sum_{k=0}^{n} T^{n+1-k} h_k.$$

It is easy to see by induction that the sequences (ρ_n) and (σ_n) are nondecreasing, and that

$$\sigma_n \leq \rho_n \leq \rho_{n+1} \leq \sigma_n + 1 \tag{3.18}$$

for all n. In particular $\rho_n \leq n$ for all n.
The following identities are easily proven

$$d_n = f_n + \sum_{k=0}^{n} T^{n-k} e_k \tag{3.19}$$

$$q_{n+1} = g_n + \sum_{k=0}^{n} T^{n-k} h_k. \tag{3.20}$$

A simple induction shows that φ_n, e_n and π_n are nonnegative for all $n \geq 0$. In fact, this is true for $n = 0$ and suppose it holds for n. It follows from the definitions that $\pi_{n+1} \geq 0$ and by the inductive hypothesis $f_n \geq 0$. Using the fact that $\{d_k\}$ is a Chacon process and equation (3.19)

$$\varphi_{n+1} = d_{n+1} - \sum_{k=0}^{n} T^{n+1-k} e_k \geq T f_n \geq 0.$$

A similar argument shows θ_n, κ_n and h_n are also nonnegative functions. Then, it follows that for all n we have

$$0 \leq e_n \leq \pi_n \leq p_0, \quad 0 \leq h_n \leq \kappa_n \leq p_0. \tag{3.21}$$

The filling scheme just defined is the same that the one introduced in [1] but with a different definition (here adapted to the Chacon processes) for the functions φ_n and θ_n. The following lemma is easily proven:

Lemma 3.22. *For all $n = -1, 0, 1, \dots,$*

$$\sum_{k=0}^{n} e_k = \rho_{n+1} p_0 + (p_0 - \pi_{n+1}) \tag{3.23}$$

and

$$\sum_{k=0}^{n} h_k = \sigma_{n+1} p_0 + (p_0 - \kappa_{n+1}). \tag{3.24}$$

We define the quantities:

$$\alpha_n = p_0 + \sum_{k=0}^{n-1}(h_k - e_k), \quad \beta_n = \sum_{k=0}^{n} e_k - \sum_{k=0}^{n-1} h_k \tag{3.25}$$

for each $n = 0, 1, \ldots$. We also let $\alpha_n = p_0$ and $\beta_n = 0$ for all negative integers n. It follows easily from the definitions that for all $n \geq 0$,

$$\alpha_n - e_n + \beta_n = p_0, \quad \alpha_{n+1} + \beta_n - h_n = p_0. \tag{3.26}$$

Also whenever $\rho_n = \sigma_n$ (and hence $\{\kappa_n = p_0\}$) using Lemma 1 and (3.26) we obtain

$$\alpha_n = \pi_n, \quad \beta_n = p_0 - (\pi_n - e_n), \quad for\, all\, n \geq 0. \tag{3.27}$$

Moreover whenever $\rho_{n+1} = \sigma_n + 1$ (and hence $\{\pi_{n+1} = p_0\}$) using Lemma 1 and equation (3.26) we obtain

$$\alpha_{n+1} = p_0 - (\kappa_n - h_n), \quad \beta_n = \kappa_n \quad for\ all\ n \geq 0. \tag{3.28}$$

Therefore the following relations are valid

$$0 \leq \alpha_n \leq p_0, \quad 0 \leq \beta_n \leq p_0 \tag{3.29}$$

for all n.

The proof that follows is essentially contained in Lemma 3 from [1]. There are some subtle differences due to our particular setting, therefore, for the sake of completeness, we present the main steps in the proof.

Lemma 3.30. *We use the notation from Definition(3.8) , let $\mathcal{U}$ be the set of all sequences $\zeta = (u_r, v_r)_{r=1,2,\ldots}$ satisfying (3.9) and (3.10). For any $\zeta \in \mathcal{U}$, let $H(\zeta)$ be the set of x such that ζ is a sequence of generalized upcrossing times at x for $\{d_r\}$, $\{q_r\}$ and $\{p_r\}$ Let $\zeta \in \mathcal{U}$, $\zeta = (u_r, v_r)_{r=1,2,\ldots,k}$.*

(i) If $r \leq k$ then,

$$r \leq \rho_{v_r+1} \text{ almost everywhere on } H(\zeta), \tag{3.31}$$

and

(ii) If $r \leq k-1$ then,

$$r \leq \sigma_{u_{r+1}+1} \text{ almost everywhere on } H(\zeta). \tag{3.32}$$

Proof. For a fixed k we will prove the result by an induction on $n \in \{-1, 0, \ldots, N\}$. The inductive hypothesis is the following:

a) Statement *i)* above holds for all r such that $v_r \le n$.

b) Statement *ii)* above holds for all r such that $u_{r+1} \le n$.

For $n = -1$ the inequalities $v_r \le n$ and $u_{r+1} \le n$ never hold. Assume that i) and ii) hold for $n = -1, 0, \ldots, m-1$, for some $m \ge 0$. We will prove that they also hold for $n = m$.

Suppose $v_r = m$, if we set $l = u_r$ then $l < m$. By the inductive hypothesis it follows that $r - 1 \le \rho_{l+1}$ almost everywhere on $H(\zeta)$. We would like first to conclude i), i.e. $r \le \rho_{m+1}$ almost everywhere on $H(\zeta)$. On the set $\{\rho_{l+1} = r-1\} \cap \{f_j = 0\}$, for any $j = l+1, \ldots, m$, we conclude from the definition of ρ_{j+1} and the fact that $r - 1 \le \rho_{l+1}$ that $r \le \rho_{j+1}$. Therefore we need only consider ρ_{m+1} on the set

$$A = \{\rho_m = r-1\} \cap \{f_{l+1} = 0\} \cap \ldots \{f_m = 0\} \cap H(\zeta).$$

Using the equations (3.11), (3.19) and (3.20) we obtain that the following inequality is valid *on the set A*

$$\sum_{j=0}^{m} p_j \le \sum_{j=l+1}^{m} \sum_{k=0}^{j} T^{j-k} e_k - \sum_{j=l+1}^{m-1} \sum_{k=0}^{j} T^{j-k} h_k. \tag{3.33}$$

Using the relationship $T^j p_0 \le p_j$, exchanging the order of sums, using the definition of α_n and β_n, and integrating over A, we have

$$\sum_{j=0}^{m} \int (T^{*j} 1_A) p_0 d\mu \le \sum_{j=0}^{m} \int (T^{*j})(\beta_{m-i} + \alpha_{l+1-i} - p_0) d\mu. \tag{3.34}$$

Given that $0 \le \alpha_n, \beta_n \le p_0$ we conclude

$$(T^{*j})\beta_{m-i} = (T^{*j}) p_0$$

almost everywhere for each $j = 0, \ldots, m$, hence $\beta_m = p_0$ almost everywhere on A. From equation (3.27) it follows that $e_m = \pi_m$, hence $\rho_{m+1} = \rho_m + 1 = r - 1 + 1$ on A. So $\rho_{m+1} \ge r$. This argument completes the proof of a). The proof of b) is analogous, we refer to reference [1] for details. $\square$

The preceeding lemma provides a very detailed description of how the filling times $\rho(x)$ and $\sigma(x)$ count the number of upcrossings and fluctuations respectively. In the next corollary we explicitly state what we are going to actually need from Lemma 3.30.

Corollary 3.35. *Almost everywhere on the set X the following inqualities hold:*

$$\omega_N(x) \le \rho_{N+1}(x) \tag{3.36}$$

$$\chi_N(x) \leq \sigma_{N+1}(x). \tag{3.37}$$

With these results we can proceed with the proof of Theorem 3.15.

Proof. Let's assume that $\gamma(D) < \infty$. From (3.19) and the fact that T is Markovian we obtain for every n,

$$(n+2)\sum_{k=0}^{n}\int e_k d\mu \leq \sum_{k=0}^{2n+1}\int d_k d\mu. \tag{3.38}$$

Therefore, using equations (3.21), (3.23) and (3.36) we conclude

$$\int \omega_N\, p_0\, d\mu \leq \int \rho_{N+1}\, p_0\, d\mu \leq \sum_{k=0}^{N}\int e_k d\mu \leq \frac{1}{N+1}\sum_{k=0}^{2N+1}\int d_k d\mu \leq\; 2\,\gamma(D)\,.$$

Similarly, if $\gamma(Q) < \infty$ we obtain:

$$\int \chi_N\, p_0\, d\mu \leq \int \sigma_{N+1}\, p_0\, d\mu \leq \sum_{k=0}^{N}\int h_k d\mu \leq \frac{1}{N+1}\sum_{k=0}^{2N+1}\int q_k d\mu \leq\; 2\,\gamma(Q)\,.$$

□

It is important to remark that the condition of T being Markovian is only used in equation (3.38). This condition is not used in any of the intermediate results which lead to Theorem 3.15 either. Under the assumption that $\{d_k\}$ is actually an additive process and T is an L^1 contraction we can derive:

$$\sum_{k=0}^{N}\int e_k d\mu \leq \int d_0 d\mu.$$

This equation and an argument similar to the given above can be used to prove an u.i. in the setting of Chacon theorem (see [5], p. 377):

Theorem 3.39. *Given two Chacon processes $\{q_k\}$, $\{p_k\}$ and an additive process $\{d_k\}$ with respect to a positive L^1 contraction T then:*
For each $N \geq 0$,

$$\int \omega_N\, p_0\, d\mu \leq \int d_0 d\mu\,. \tag{3.40}$$

Corollary 3.41. *Consider the ergodic averages*

$$R_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} T^k d_0$$

where T is a positive linear contraction satisfying $T1 \leq 1$ and in Definition 3.15 take $\{d_k = T^k d_0\}$, $\{q_k = \alpha\}$ and $\{p_k = \eta\}$ equation (3.17) becomes

$$\int \chi_N d\mu \leq \frac{\alpha\mu(X)}{\eta}. \tag{3.42}$$

Whenever $\mu(X)$ is finite, equation (3.42) gives the surprising result that for a fixed η we get a uniform bound in α independently of d_0.

4 Bishop's Lemma

In this section we will use Theorem 3.15 to stablish Lemma 1 from [2]. This is an interesting combinatorial inequality which, as proven in [2], implies Lebesgue's differentiation theorem, the ergodic theorem for measure preserving transformations and Doob's martingale theorem. We will also indicate how an improved version of the upcrossing inequalities presented in the previous section implies integral bounds for the oscillations of ergodic averages. Oscillations in Ergodic Theory have recently received due attention in [6]. Let us consider the situation described in Bishop first. Here we are given real numbers $\alpha < \beta$ and a set of points $\Gamma \subset \mathbf{R}^2$, $\Gamma = \{x^1, y^1, \ldots, x^n, y^n\}$, where Γ is *elementary.* In particular this means that

$$x_2^{i+1} - \alpha x_1^{i+1} \leq y_2^i - \alpha y_1^i \tag{4.1}$$

for $i = 1, \ldots, n-1$. Moreover, the following also holds for an elementary set Γ

$$x_1^i = y_1^i, \quad x_2^i < y_2^i \;\; for \;\; i = 1, \ldots, n.$$

Below we define the counting function considered by Bishop.

Definition 4.2. *Given real numbers α, β as above, an elementary set Γ , $z = (z_1, z_2) \in \mathbf{R}^2$ and $t \in \mathbf{R}$. Then define*

$$\omega(z, \alpha, \beta, \Gamma) = \sup\{k : \exists x^1, \;\; y^1, \ldots, x^k, \;\; y^k \in \Gamma \; with \; property \; P(z, \alpha, \beta)\}.$$

Where the points $x^1, y^1, \ldots, x^k, y^k \in \Gamma$ are said to have the property $P(z, \alpha, \beta)$ if

$$z_1 < x_1^1, \; x_2^i - \alpha x_1^i < z_2^i - \alpha z_1^i, \; and \; z_2^i - \beta z_1^i < y_2^i - \beta y_1^i, \; for \; all \; 1 \leq i \leq k.$$

Finally define:

$$\omega(t, \alpha, \beta, \Gamma) = \sup\{\omega(z, \alpha, \beta, \Gamma) : z_1 = t\} \tag{4.3}$$

Theorem 4.4. *If Γ is elementary*

$$\int \omega(t, \alpha, \beta, \Gamma) dt \leq (\beta - \alpha)^{-1} \sum_1^n (y_2^i - x_2^i).$$

Proof. Let

$$B = \{x_1^1, \dots, x_1^n\}. \tag{4.5}$$

Let $\psi = \{u^1, v^1, \dots, u^k, v^k\} \subset \Gamma$, where $u_1^i \le v_1^i$ for all i and $v_1^i < u_1^{i+1}$ for all $i = 1, \dots, k-1$. Then we will say that ψ is an admissible sequence of length k. For any $t \in \mathbf{R}$ we will say that ψ is t-admissible if $t < u_1^1$ and if for all $i, j \in \{1, \dots, k\}$ we have

$$(u_2^i - \alpha u_1^i) - (v_2^j - \beta v_1^j) < (\beta - \alpha)t. \tag{4.6}$$

Define the function κ on $\mathbf{R}$ by

$$\kappa(t) = \sup\{\text{length } \psi \ : \ \psi \text{ is } t\text{−admissible}\}. \tag{4.7}$$

It is easy to see that

$$\kappa(t) \ge \omega(t, \alpha, \beta, \Gamma),$$

where $\omega(t, \alpha, \beta, \Gamma)$ is the quantity defined above.
We are required to prove:

$$\int \kappa \, dt \le 1/(\beta - \alpha) \sum_{i=1}^{n} (y_2^i - x_2^i). \tag{4.8}$$

Suppose that $\delta_k > 0$, $L_k = \delta_k \mathbf{Z}$. Clearly

$$\int \kappa dt = \lim_{k \to \infty} \delta_k \sum_{t \in L_k} \kappa(t - \delta_k). \tag{4.9}$$

By an approximation argument we can easily reduce the proof to the case where for some $\delta > 0$ we have $B \subset \delta\mathbf{Z}$. Thus we can choose $\delta_k \searrow 0$ such that $B \subset \delta_k \mathbf{Z}$ for all k. Thus to prove Lemma 1 of [2] it is enough to show for any $\delta > 0$ with $B \subset \delta\mathbf{Z}$ we have

$$\delta \sum_{t \in L - B} \kappa(t - \delta) \le 1/(\beta - \alpha) \sum_{i=1}^{n} (y_2^i - x_2^i). \tag{4.10}$$

Fix δ, and let $\tau \ : \ L \to L$ be defined by $\tau(t) = t + \delta$. Define T on functions by $Tf = f \circ \tau$. Let J be a finite interval of L containing B, and large enough so that J contains any point t in L with $\kappa(t) > 0$. (It is easy to see that such an interval exists). Let

$$p = (\beta - \alpha)\delta \mathbf{1}_J. \tag{4.11}$$

Let

$$f(x_1^i) = y_2^i - x_2^i, \tag{4.12}$$

for $i = 1, \dots, n$, and let $f(t) = 0$ for all other $t \in L$. Let

$$g(x_1^i) = y_2^i - x_2^{i+1} + \alpha(x_1^{i+1} - x_1^i), \tag{4.13}$$

for $i = 1, \ldots, n-1$, and let $g(t) = 0$ for all other $t \in L$. Let $t \in L - B$ and let $\psi = (u^i, v^i)_{i=1,\ldots,k}$ be $(t-\delta)$-admissible. For each $i = 1, \ldots, k$, there exists m_i such that either $u^i = x^{m_i}$ or $u^i = y^{m_i}$. After considering (4.6) for a moment we notice that we can modify ψ, if necessary, to ensure that $u^i = x^{m_i}$. Similarly we can ensure that for each $j = 1, \ldots, k$ there exists some ℓ_j such that $v^j = y^{\ell_j}$. We would like to show that the quantities T, f, g, p have at least k upcrossings at x. To do that we have to construct an upcrossing sequence in the sense of Section 3. We can't work with the letters u and v because we have used them when re-writing Bishop's definition (i.e Definition 4.2), therefore, we will call our times ξ_r and η_r. Let ξ_r be that nonnegative integer such that

$$\tau^{\xi_r+1}(t) = u_1^r = x_1^{m_r}, \tag{4.14}$$

for $r = 1, \ldots, k$.

Let η_r be that positive integer such that

$$\tau^{\eta_r}(t) = v_1^r = x_1^{\ell_r}, \tag{4.15}$$

for $r = 1, \ldots, k$.

Let

$$x_1^q = \inf\{x_1^i \ : \ t < x_1^i\}. \tag{4.16}$$

Now consider

$$\sum_{j=0}^{\xi_r} T^j(f-g)(t) = \sum_{q \le i < m_r} (f-g)(x_1^i).$$

This is

$$\sum_{q \le i < m_r} [(y_2^i - x_2^i) - (y_2^i - x_2^{i+1}) - \alpha(x_1^{i+1} - x_1^i)] = x_2^{m_r} - x_2^q - \alpha(x_1^{m_r} - x_1^q).$$

Hence we have shown that

$$\sum_{j=0}^{\xi_r} T^j(f-g)(t) = u_2^r - \alpha u_1^r - (x_2^q - \alpha x_1^q). \tag{4.17}$$

We also want to consider

$$\sum_{j=0}^{\eta_r-1} T^j(f-g-p)(t) + T^{\eta_r}(f-p)(t) = \sum_{j=0}^{\eta_r-1} T^j(f-g)(t) + T^{\eta_r} f(t) - \sum_{j=0}^{\eta_r} T^j p(t).$$

This is

$$\sum_{q \le i < \ell_r} (f-g)(x_1^i) + f(x_1^{\ell_r}) - (\beta-\alpha)(x_1^{\ell_r} - t + \delta) = \tag{4.18}$$
$$x_2^{\ell_r} - x_2^q - \alpha(x_1^{\ell_r} - x_1^q) + y_2^{\ell_r} - x_2^{\ell_r} - (\beta-\alpha)(x_1^{\ell_r} - t + \delta).$$

Hence we have shown that

$$(4.19)\quad \sum_{j=0}^{\eta_r-1} T^j(f-g-p)(t) + T^{\eta_r}(f-p)(t) = v_2^r - \beta v_1^r - (x_2^q - \alpha x_1^q) + (\beta-\alpha)(t-\delta).$$

Using these equations and (4.6) (with t replaced by $t-\delta$) it follows easily that $\xi_1, \eta_1, \ldots, \xi_k, \eta_k$ is a generalized upcrossing sequence at t. Hence we have shown that for any $t \in L - B$ we have

$$(4.20)\qquad \kappa(t-\delta) \leq \omega(t),$$

where ω is the function defined in our paper rather than one of the functions in Bishop. By Theorem 3.15 from Section 3 we then have

$$(4.21)\qquad \sum_{t\in L-B} \kappa(t-\delta)p(t) \leq \sum_{t\in L} f(t),$$

and this is (4.10). □

We will end this section by presenting a simple argument that links the notion of upcrossings to *oscillations* (see definition below). For the rest of this section we use the following notation and assumptions: for $f \in L^1(X)$ (X a finite measure space), $A_n f(x) = \frac{1}{n+1}\sum_{k=0}^{n} T^k f(x)$ denotes the Cesaro averages for a positive linear contraction T satisfying $T1 \leq 1$. We also define $A_{-1}f(x) = 0$ for all x.

Definition 4.22. *For a fixed integer $N > 0$, a real number $\eta > 0$ and $x \in X$, define*

$$z_{N,\eta}(x) = \sup\{k : \xi = (t_r)_{r=1,\ldots,k}.\}$$

where ξ satisfies:

$$-1 \leq t_1 < t_2 \ldots < t_k \leq N$$

and

$$(4.23)\qquad |A_{t_{r+1}}f(x) - A_{t_r}f(x)| \geq \eta, \quad \textit{for all } r = 1, \ldots, k-1$$

also define

$$z_\eta(x) = \sup\{z_{N,\eta}(x) : N > 0\}$$

the function z_η will be called the oscillation function (also called η-jumps, see [6]).

We will use a result on upcrossing inequalities established by Bishop in [3] (Theorem 6 and Corollary pp. 236-238, see also [4]). Before stating this theorem, we introduce some notation. The function $w_{N,\eta,\alpha}(x)$ in the theorem below is defined as follows: for $f \in L^1$, an integer $N > 0$, a real number α, and a real number $\eta > 0$ define

$$(4.24)\qquad w_{N,\eta,\alpha}(x) = \sup\{k : \zeta = (u_r, v_r)_{r=1,\ldots,k}.\}$$

where the sequence ζ satisfies,

$$-1 \leq u_1 < v_1 < u_2 < \ldots < v_k \leq N \tag{4.25}$$

$$A_{u_r} f(x) \leq \alpha \ \ \textit{and} \ \ A_{v_r} f(x) \geq (\alpha + \eta) \tag{4.26}$$

for $r = 1, \ldots, k$.

Bishop's theorem is:

Theorem 4.27. *For N, α, η and f as above:*

$$\int w_{N,\eta,\alpha} d\mu \leq \frac{1}{\eta} \int (f - \alpha)_+ d\mu$$

Remark

Theorem 4.27 is true in a more general setting, in fact the upcrossing function considered by Bishop is an upper bound for the function defined above. We also remark that our *generalized upcrossing function* from the previous section is also an upper bound for the function $w_{N,\eta,\alpha}$; however Theorem 3.39 is not strong enough to establish Theorem 4.28 below.

The following theorem generalizes results from [6] to the operator case.

Theorem 4.28. *Let $f \in L^2(X)$, $f \geq 0$ and a real number $\eta > 0$, then:*

$$\int z_\eta d\mu \leq \frac{8}{\eta^2} \int f^2 d\mu. \tag{4.29}$$

Proof. We will prove:

$$\int z_{N,\eta} d\mu \leq \frac{8}{\eta^2} \int f^2 d\mu$$

Define $\alpha_i = i\ \frac{\eta}{2}$ for $i = 1, \ldots$. Also define:

$$w_{N,\frac{\eta}{2}}(x) = \sum_{i=0}^{\infty} w_{N,\frac{\eta}{2},\alpha_i}(x).$$

We now make the key observation that for all x where the quantities are defined:

$$z_{N,\eta}(x) \leq 2\ w_{N,\frac{\eta}{2}}(x). \tag{4.30}$$

Using Theorem 4.27 we compute:

$$\sum_{i=1}^{\infty} \int w_{N,\frac{\eta}{2},\alpha_i} d\mu \leq \frac{2}{\eta} \sum_{i=1}^{\infty} \int_{\{f > \alpha_i\}} (f - \alpha_i) d\mu \leq \frac{2}{\eta} \int \sum_{i=1}^{i < \frac{2f}{\eta}} (f - i\frac{\eta}{2}) d\mu \leq \frac{4}{\eta^2} \int f^2 d\mu.$$

Hence by Fubini's theorem,

$$\int z_{N,\eta}(x) d\mu \leq \int 2\ w_{N,\frac{\eta}{2}} d\mu \leq \frac{8}{\eta^2} \int f^2 d\mu.$$

$\square$

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